Motion and Stability of a Rotating Space Station-Cable-Counterweight Configuration

PERICLES STABEKIS* AND PETER M. BAINUM†
Howard University, Washington, D.C.

An analysis of the dynamics of a rotating, connected two-body satellite system is presented. The motion is restricted to the orbital plane in which the system mass center follows a circular orbit. The cable is assumed to be extensible, but of negligible mass. The Lagrangian equations of motion are developed and linearized about the equilibrium motion, where the cable tensile force is balanced by the centrifugal force. Necessary and sufficient Routh-Hurwitz stability criteria are obtained for various special cases. Damping system parameters are optimized by repeated examination of the roots of the system characteristic equation. Minimum time constants are found to be insensitive to small changes in the amount of cable damping, but are dependent on the amount of rotational damping of the end body motions. Spin axis drift attributed to gravity-gradient torques is examined quantitatively and found to be small.

Nomenclature

A_j = amplitude coefficient factors for small oscillation modes
a_{ij} = constant coefficient in the linearized equations of motion
c_i = rotational spring constants $i = 1,2$
GM_E = gravitational constant for Earth
I_{yi} = moment of inertia of body i about its axis of rotation, $i = 1,2$
k = cable elastic (Hooke's Law) constant
k_i = rotational damping constant, $i = 1,2$
k_3 = cable damping constant
l = cable length between the two attachment points
l_0 = unstressed cable length
m_i = mass $i = 1,2$
N = number of degrees of freedom in the system
Q_j = generalized (nonconservative) applied forces
$q_j,\dot{q}_j$ = generalized coordinates, velocities
$\mathbf{R}$ = position vector from geocenter to system center of mass
R = magnitude of $\mathbf{R}$
$\mathbf{r}_i$ = position vector from the system c.m. to the c.m. of body i, $(i = 1,2)$
s = nominal spin rate of the system
T = system kinetic energy
t = time
V = system potential energy
β = variational coordinate describing rotation of the system
θ = generalized coordinate describing the inertial rotation of the system
μ = reduced mass of system
ρ_i = attachment radius, $i = 1,2$
τ = nondimensional time ($\tau = st$)
φ_i = generalized coordinate describing rotation, $i = 1,2$
ω = orbital angular velocity
ω_{yi} = $\omega + \dot{\theta} + \dot{\varphi}_i$, $i = 1,2$
$\mathfrak{F}$ = Rayleigh dissipation function

Subscripts

1,2 = space station and counterweight, respectively

Superscripts

$(\dot{\ })$ = $d(\ )/dt$
$(\ )'$ = $d(\ )/d\tau$

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* Staff Engineer, Exotech Systems Inc., Washington, D.C.; formerly graduate student, Department of Mechanical Engineering, Howard University.

† Associate Professor of Aerospace Engineering; also Consulting Engineer, The Johns Hopkins University, Applied Physics Laboratory, Silver Spring, Md. Member AIAA.

Introduction

THE motion and stability of a rotating space station-cable-counterweight system is the subject of this analysis. The configuration considered consists of the manned module (station) connected by an extensible cable to a counterweight which could be the final stage of the boost vehicle (Fig. 1). The motivation for this investigation was provided by Ref. 1, which treated a similar case but lacked a rigorous stability analysis. Based on a limited number of numerical cases, the feasibility of using damping proportional to the cable extensional rate in conjunction with rotational damping on both bodies was demonstrated. It was also shown that when either cable damping or rotary damping was removed, instability resulted.[1]

Other treatments studying the dynamic behavior of similar connected two body configurations include the papers by Paul[2] and Chobotov.[3] Both assumed the end masses to be point masses. Paul considered the planar motion and stability of a gravity-gradient-stabilized, extensible dumbbell satellite system where the mass effects of the cable were neglected. It was shown that, if the internal friction arised from "material damping" within the spring, there will be relatively little damping of a viscous nature but that there is a nonlinear time-independent type of hysteretic damping which could be significant.[2] Chobotov[3] included the mass and elasticity of the cable for a rotating system. It was found that the gravity-gradient effects upon the small amplitude vibration stability of the rotating cable-counterweight system are very small and that the stability criteria are functions of the cable natural frequencies, the angular velocities of the space station and orbital motion, and viscous damping. Bainum et al[4] considered the effects of distributed end masses for the case of a gravitationally stabilized, nonrotating satellite system and concluded that a combination of tether system damping and rotational damping of the motion of the end masses about their own mass centers must be employed; the use of one damping scheme without the other will not provide adequate damping of each normal mode of oscillation for the relatively long (e.g., 3 naut miles) connecting lengths considered.

Robe[5] dealt with the motion and stability of two identical but unsymmetrical rigid bodies connected by a single tether—a gravitationally stabilized nonrotating system. This three-dimensional analysis showed that there is a decoupling of the small-amplitude motions in the orbital plane from those outside the plane.

The present work considers both a rotating system and distributed end masses. One of its possible applications is to induce a desired artificial gravitational field on the manned module and thus to overcome possible adverse physiological effects arising from a zero g environment. The motion is restricted to the orbital plane with the system's mass center assumed to follow a circular orbit. The cable's mass is neglected, and external perturbations (e.g., solar pressure and thermal bending) are not taken into account.

Development of Equations of Motion

The rotational equations of motion will be developed by using the general Lagrangian formulation for a nonconservative system as derived directly from D'Alembert's principle,[6] $d/dt(\partial T/\partial \dot{q}_j) - \partial T/\partial q_j = Q_j$.

As a result of the previously stated assumptions, it can be seen that there are four degrees of freedom for the system (Fig. 1) viz., 1) the length of the connecting cable can change; 2) the centerline connecting the mass center of the manned section and the counterweight can rotate relative to some reference direction such as the local vertical; and 3) the manned section and the counterweight each can rotate about an axis through its own mass center, normal to the orbital plane. The corresponding generalized coordinates used in describing the motion are, $q_1 = l$, $q_2 = \theta$, $q_3 = \varphi_1$, and $q_4 = \varphi_2$.

Kinetic Energy

The total kinetic energy of the system of two masses can be expressed,

$$T = \tfrac{1}{2}I_{y_1}\omega_{y_1}{}^2 + \tfrac{1}{2}I_y\,\omega_{y_2}{}^2 + \tfrac{1}{2}m_1|\dot{\mathbf{R}} + \dot{\mathbf{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\mathbf{R}} + \dot{\mathbf{r}}_2|^2 \quad (1)$$

where the symbols are defined in the nomenclature section. The velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ can be obtained from the following. From consideration of the definition of the system center of mass,

$$(m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2) = \mathbf{O} \quad (2)$$

If the difference between the velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ is $\mathbf{u}$, i.e., $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{u}$, then

$$\dot{\mathbf{r}}_1 = m_2\mathbf{u}/(m_1 + m_2) \text{ and } \dot{\mathbf{r}}_2 = m_1\mathbf{u}/(m_1 + m_2) \quad (3)$$

In order to develop an expression for $\mathbf{u}$ the following vector quantities will be introduced. Let $\boldsymbol{\rho}_1$ be a vector directed along attachment arm ρ_1 from the attachment point to the c.m. of m_1, $\boldsymbol{\rho}_2$ be a vector directed along attachment arm ρ_2 from the attachment point to the c.m. of m_2, $\mathbf{l}$ be a vector directed along the tether line from the attachment point of the lower body to the attachment point of the upper body such that $\mathbf{l} = l\mathbf{a}$ where $\mathbf{a}$ is a unit vector, and $\mathbf{j}$ be a unit vector along the y axis (perpendicular to the orbital plane). It is apparent that

$$[(\mathbf{r}_1 - \boldsymbol{\rho}_1) - (\mathbf{r}_2 - \boldsymbol{\rho}_2)]^{\cdot} = (l\mathbf{a})^{\cdot} = \dot{l}\mathbf{a} + \Omega\mathbf{j} \times l\mathbf{a} \quad (4)$$

where $\Omega = \omega + \dot{\theta}$ and

$$\dot{\boldsymbol{\rho}}_1 = \omega_{yi}\mathbf{j} \times \boldsymbol{\rho}_i,\ i = 1,2 \quad (5)$$

From Eqs. (4) and (5)

$$\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 = \dot{l}\mathbf{a} + \mathbf{j} \times [\Omega l\mathbf{a} + \omega_{y1}\boldsymbol{\rho}_1 - \omega_{y2}\boldsymbol{\rho}_2] = \mathbf{u} \quad (6)$$

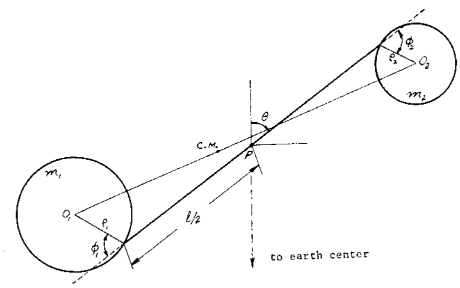

Fig. 1 Primary configuration considered in present analysis.

By substitution of Eqs. (2, 3, and 6) into Eq. (1)

$$T = \tfrac{1}{2}I_{y_1}\omega_{y_1}{}^2 + \tfrac{1}{2}I_y\,\omega_{y_2}{}^2 + \tfrac{1}{2}(m_1 + m_2)R^2\omega^2 + \tfrac{1}{2}\mu u^2 \quad (7)$$

where $\mu = m_1m_2/(m_1 + m_2)$, the system reduced mass, and u^2 can be expanded using Eq. (6) to yield,

$$u^2 = (\dot{l} - \omega_{y_1}\rho_1 \sin\varphi_1 - \omega_{y_2}\rho_2 \sin\varphi_2)^2 + (\Omega l + \omega_{y_1}\rho_1 \cos\varphi_1 + \omega_{y_2}\rho_2 \cos\varphi_2)^2 \quad (8)$$

It should be noted that the third term in Eq. (7) is a constant for the case of a circular orbit and does not affect the rotational equations of motion.

Potential Energy

The terms included in the potential energy are as follows: 1) gravity-gradient terms arising from the fact that the masses of the manned station and the counterweight each have a distributive mass and that their mass centers are displaced from the system's mass center; 2) the potential energy of the system with the total mass assumed to be concentrated at the system mass center with the system moving in the orbital plane; 3) the potential energy associated with generalized restoring forces inherent in the elastic cable and the end body damping systems.

In the present analysis, the gravity-gradient term will be omitted since the quantitative drift effects of the gravity-gradient torques (though small) will be treated in a later section.

The component of the potential energy attributed to the orbital motion of the combined mass of the system is given by,

$$V_1 = GM_E(m_1 + m_2)/R \quad (9)$$

This term will not have any effect on the rotational equations of motion for the case of circular orbital motion.

The tension due to the expansion of the cable is assumed to be linearly related to the extension from an unstressed length, l_0, in accordance with the Hooke's Law,[7]

$$V_2 = \tfrac{1}{2}k(l - l_0)^2 \quad (10)$$

such that

$$\partial V/\partial l = k(l - l_0) \quad (11)$$

is the generalized tension force. In Expressions (10) and (11), k has the units of force per unit length and may be considered as an equivalent linear restoring spring constant. This restoring effect is essential since without it, the end masses, after deployment, would tend to drift into separate orbits determined by the terminal conditions of deployment.

Another potential energy term results from considering a rotational damping, that is, damping of the rotational motions of both the space station and the counterweight. Such damping could provide a restoring torque, and it might be implemented by using the interaction of on-board magnetics with the ambient magnetic field of the earth.[4] A satellite which has a large magnetic dipole moment will tend to align itself along the local direction of the magnetic field.

The resulting restoring torque can be derivable from the following potential term,

$$V_3 = \tfrac{1}{2}(c_1\varphi_1^2 + c_2\varphi_2^2) \tag{12}$$

Rayleigh Dissipation Function

The Rayleigh dissipation function may be used for nonconservative systems involving only equivalent linear viscous damping. The Rayleigh function is explicitly a function of the generalized velocities, ($\dot{q}_j$), such that the resulting dissipative forces are derivable treating $\mathfrak{F}$ as a "potential type" term,[6]

$$Q_j = -\partial\mathfrak{F}/\partial\dot{q}_j \tag{13}$$

For this case it has been assumed that

$$\mathfrak{F} = \tfrac{1}{2}(k_1\dot{\varphi}_1^2 + k_2\dot{\varphi}_1^2 + k_3\dot{l}^2) \tag{14}$$

yielding dissipative forces in three of the generalized coordinates. Here k_1 and k_2 are the equivalent viscous rotary damping constants for the station and the counterweight, and k_3 is the equivalent viscous damping constant for the cable. In practice such damping can be realized from mechanical systems with viscous damping or from magnetic damping systems with energy losses attributed to eddy current damping.

Lagrange's General Equation of Motion

Lagrange's general equation of motion as applicable to a nonconservative system with linear viscous damping may be written,[8]

$$(d/dt)(\partial T/\partial\dot{q}_j) - \partial(T - V)/\partial q_j = -\partial\mathfrak{F}/\partial\dot{q}_j \tag{15}$$

where T and V represent the total kinetic and potential energies of the system. For this application $N = 4$ degrees of freedom.

It should be noted that from the expressions of the total kinetic and potential energies as well as of the Rayleigh dissipation function ($\mathfrak{F}$) it is apparent that the Lagrangian $(T - V)$, and $\mathfrak{F}$ do not contain the angle θ explicitly, thus leading to a first integral for this particular generalized coordinate (p_θ is conserved).

In expanding Lagrange's general equations of motion the following small-angle assumption will be made. Products of angles and products of angle and rate or angle and acceleration are regarded as higher order terms when compared with (nonlinear) terms involving products of lengths and angles or lengths and angular rates. These higher order terms will be excluded.

Eliminating the laborious derivation, and letting $L \equiv l + \rho_1 + \rho_2$, and $\gamma \equiv \omega + \dot{\theta}$, the equations of motion become,

l Equation

$$\mu\ddot{l} - \mu\rho_1\dot{\varphi}_1^2 - \mu\rho_2\dot{\varphi}_2^2 - \mu L\gamma^2 + k(l - l_0) + k_3\dot{l} = 0 \tag{16}$$

θ Equation

$$I_{y_1}(\ddot{\theta} + \ddot{\varphi}_1) + I_{y_2}(\ddot{\theta} + \ddot{\varphi}_2) + \mu L^2\ddot{\theta} + 2\mu L\gamma\dot{l} = 0 \tag{17}$$

φ_1 Equation

$$I_{y_1}(\ddot{\theta} + \ddot{\varphi}_1) + \mu\rho_1\{L\ddot{\theta} + 2\gamma\dot{l} + \rho_1\ddot{\varphi}_1 + \rho_2\ddot{\varphi}_2 + \rho_2\dot{\varphi}_1^2(\varphi_1 - \varphi_2)\} + \rho_1\mu\gamma^2\{l\varphi_1 + \rho_2(\varphi_1 - \varphi_2)\} + c_1\varphi_1 + k_1\dot{\varphi}_1 = 0 \tag{18}$$

φ_2 Equation

$$I_{y_2}(\ddot{\theta} + \ddot{\varphi}_2) + \mu\rho_2\{L\ddot{\theta} + 2\gamma\dot{l} + \rho_2\ddot{\varphi}_2 + \rho_1\ddot{\varphi}_1 - \rho_1\dot{\varphi}_2^2(\varphi_1 - \varphi_2)\} + \rho_2\mu\gamma^2\{l\varphi_2 - \rho_1(\varphi_1 - \varphi_2)\} + c_2\varphi_2 + k_2\dot{\varphi}_2 = 0 \tag{19}$$

The first integral, $p_\theta = \partial T/\partial\dot{\theta}$, is given by

$$I_{y_1}(\gamma + \dot{\varphi}_1) + I_{y_2}(\gamma + \dot{\varphi}_2) + \mu\gamma L^2 = \text{const (1)} \tag{20}$$

The l Equation will now be examined for near equilibrium conditions. At equilibrium, $\varphi_1 = \varphi_2 = 0$, $\ddot{l} = \dot{l} = 0$, and the rate of spin is identically the nominal spin rate of the system, s. Thus, the l Equation provides the following equilibrium condition:

$$\mu(l_{eq} + \rho_1 + \rho_2)(\omega + s)^2 = k(l_{eq} - l_0) \tag{21}$$

Thus, there should be a balance between the tensile and centrifugal forces in the rotating frame when the system is at equilibrium. (A numerical evaluation of the equilibrium conditions is discussed later.)

Since the stability of motion about the equilibrium point is of special interest, the equations of motion will now be linearized about that particular solution. For this purpose the following substitutions are used:

$$\bar{l} = (l - l_{eq})/l_{eq}$$

$$\theta = st + \beta \;(\dot{\theta} = s + \dot{\beta},\; \ddot{\theta} = \ddot{\beta})$$

$$\tau = st \;(\text{i.e. } \dot{\beta} = s\beta',\; \ddot{\beta} = s^2\beta'', \text{ etc.})$$

where the primes now denote derivatives with respect to the nondimensional time, τ, and where $\bar{l}$, β, and τ are nondimensional quantities as are $\bar{I}_{y_1}$, $\bar{\rho}_1$, $\bar{k}_3$, etc. (see Appendix).

Letting $\bar{L} = 1 + \bar{\rho}_1 + \bar{\rho}_2$, and $\Gamma = (\omega/s) + 1$, the linearized equations of motion in the new variational coordinates $\bar{l}$, β, φ_1, φ_2 are,

$$\bar{l}'' + \bar{k}_3\bar{l}' - \bar{l}(\Gamma^2 + \bar{k}) - 2\Gamma\bar{L}\beta' = 0 \tag{22}$$

$$\beta''(\bar{I}_{y_1} + \bar{I}_{y_2} + \bar{L}^2) + \bar{I}_{y_1}\varphi_1'' + \bar{I}_{y_2}\varphi_2'' + 2\Gamma\bar{L}\bar{l}' = 0 \tag{23}$$

$$[\bar{I}_{y_1} + \bar{\rho}_1^2]\varphi_1'' + \bar{k}_1\varphi_1' + [\bar{\rho}_1\Gamma^2(1 + \bar{\rho}_2) + c_1]\varphi_1 + [\bar{I}_{y1} + \bar{L}\bar{\rho}_1]\beta'' + 2\Gamma\bar{\rho}_1\bar{l}' + \bar{\rho}_1\bar{\rho}_2\varphi_2'' - \bar{\rho}_1\bar{\rho}_2\Gamma^2\varphi_2 = 0 \tag{24}$$

$$[\bar{I}_{y_2} + \bar{\rho}_2^2]\varphi_2'' + \bar{k}_2\varphi_2' + [\bar{\rho}_2\Gamma^2(1 + \bar{\rho}_1) + c_2]\varphi_2 + [\bar{I}_{y_2} + \bar{\rho}_2\bar{L}]\beta'' + 2\Gamma\bar{\rho}_2\bar{l}' + \bar{\rho}_1\bar{\rho}_2\varphi_1'' - \bar{\rho}_1\bar{\rho}_2\Gamma^2\varphi_1 = 0 \tag{25}$$

The first integral becomes, after linearization, at equilibrium,

$$\Gamma\{(\bar{I}_{y1} + \bar{I}_{y2}) + \bar{L}^2\} = \text{const (2)} \tag{26}$$

Also, the fact that such a first integral exists indicates a redundancy in the system meaning that only three of the four selected variational coordinates are independent. As a result, there are only three independent linearized variational equations of motion since, for example, β' and β'' can be now expressed in terms of the other generalized coordinates.

The equilibrium condition, given by Eq. (21) becomes after linearization,

$$\bar{L}\Gamma^2 = \bar{k}(1 - \bar{l}_0) \tag{27}$$

Substituting the values for β' and β'' in the l, φ_1, and φ_2 equations yields the following independent variational equations,

l Equation

$$\bar{l}'' + a_5\bar{l}' + (a_6 + a_4^2/a_1 + a_{18}) + (a_2a_4/a_1)\varphi_1' + (a_3a_4/a_1)\varphi_2' = 0 \tag{28}$$

φ_1 Equation

$$(a_7 - a_2a_{10}/a_1)\varphi_1'' + a_8\varphi_1' + a_9\varphi_1 + (a_{11}a_{10}a_4/a_1)\bar{l}' + (a_{12} - a_3a_{10}/a_1)\varphi_2'' + a_6a_{12}\varphi_2 = 0 \tag{29}$$

φ_2 Equation

$$(a_{13} - a_3a_{16}/a_1)\varphi_2'' + a_{14}\varphi_2' + a_{15}\varphi_2 + (a_{17} - a_4a_{16}/a_1)l' + (a_{12} - a_2a_{16}/a_1)\varphi_1'' + a_6a_{12}\varphi_1 = 0 \quad (30)$$

where the constants $a_1 \ldots a_{18}$ correspond to the coefficients of the linear terms (see Appendix). Equations (28–30) are the complete linearized equations of motion and will be used for the stability analysis of the linear system.

Stability Analysis

Criterion

We use the criterion of Routh-Hurwitz,[9] according to which all the roots of the real polynomial $f(z) = a_0z^n + a_1z^{n-1} + \ldots + a_n(a_0 > 0)$ have negative real parts if and only if the inequalities

$$a_1 > 0, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0$$

$$\ldots \begin{vmatrix} a_1 & a_3 & a_5 \ldots 0 \\ a_0 & a_2 & a_4 \ldots 0 \\ 0 & a_1 & a_3 \ldots 0 \\ 0 & a_0 & a_2 \ldots 0 \\ \cdot & \cdot & \cdot \ldots 0 \\ \cdot & \cdot & \cdot \ldots a_n \end{vmatrix} > 0$$

hold. This criterion is a necessary and sufficient condition for stability. The Routhian determinants from first order to order $2N$ (where N is the number of degrees of freedom) must have the same sign when expanded and evaluated, although only half of these are actually required.[10]

Application

Expansion of system characteristic equation

For small oscillations it is assumed that $\bar{l} = A_1e^{\lambda\tau}$, $\varphi_1 = A_2e^{\lambda\tau}$, $\varphi_2 = A_3e^{\lambda\tau}$ where λ, in general, is a complex number. Substituting back into Eqs. (28–30) yields a set of three linear equations for the A_j $(j = 1 \rightarrow 3)$. These equations will have a nontrivial solution for the A_j if the determinant of the linearized system matrix vanishes,

$$\begin{vmatrix} \lambda^2 + a_5\lambda + a_6 + \dfrac{a_4^2}{a_1} + a_{18} & \dfrac{a_2a_4}{a_1}\lambda & \dfrac{a_3a_4}{a_1}\lambda \\ a_{11}\lambda - \dfrac{a_4a_{10}}{a_1}\lambda & a_7\lambda^2 - \dfrac{a_2a_{10}}{a_1}\lambda^2 + a_8\lambda + a_9 & a_{12}\lambda^2 - \dfrac{a_3a_{10}}{a_1}\lambda^2 + a_6a_{12} \\ a_{17}\lambda - \dfrac{a_4a_{16}}{a_1}\lambda & a_{12}\lambda^2 - \dfrac{a_2a_{16}}{a_1}\lambda^2 + a_6a_{12} & a_{13}\lambda^2 - \dfrac{a_3a_{16}}{a_1}\lambda^2 + a_{14}\lambda + a_{15} \end{vmatrix} = 0 \quad (31)$$

The characteristic equation $f(\lambda)$ is obtained by expanding the determinant in Eq. (31). After considerable algebraic manipulation, the characteristic equation is found to be a sixth order polynomial in λ with all powers of λ and a constant term present.[11]

A necessary condition for stability is that the coefficients occurring in $f(\lambda)$ all must have the same sign and none of them should be zero.

Examining the sign of the coefficient of each term of the characteristic equation, it was found that the coefficient of λ^6 can be reduced to:

$$\bar{I}_{y_1}\bar{I}_{y_2}[\bar{L} + (\bar{\rho}_1 + \bar{\rho}_2)^2] + \bar{I}_{y_1}\bar{\rho}_2^2(1 + \bar{\rho}_2^2 + \bar{\rho}_1\bar{\rho}_2 + \bar{\rho}_1 + 2\bar{\rho}_2) + \bar{\rho}_1^2\bar{I}_{y_2}\bar{L}^2 \quad (32)$$

As it is readily apparent from the above expression, the constant term of the characteristic equation is given by[11] $(a_9a_{15} - a_6^2a_{12}^2)(a_4^2 + a_1a_6 + a_1a_{18})$. Substituting the values for the a's and carrying out certain algebraic manipulations yields the following conditions for the constant term to be positive,

$$c_1 \text{ and } c_2 \text{ must be positive} \quad (33)$$

and

$$\bar{k} > \Gamma^2(a_1 - 4\bar{L}^2)/a_1 \quad (34)$$

It was found that the constant term vanishes when $\Gamma = 0$ and $c_1 = c_2 = 0$. This fact indicates that when the system is spinning at a rate equal and opposite to the orbital angular velocity with no rotary restoring constant present instability results. (However, general conclusions regarding this should be made only after consideration of more complete gravitational effects than those included here.)

It can be similarly shown that the remaining coefficients of the characteristic equation are positive, provided that $\bar{k}_3$ is positive, and $\bar{k}_1$, $\bar{k}_2 > 0$. Thus, the examination of the necessary condition for stability has been completed.

Special case of complete symmetry

The application of the Routh-Hurwitz necessary and sufficiency criterion as well as a complete paper analysis of the system characteristic equation is virtually impossible due to its complexity. Therefore, the Routh-Hurwitz criterion will be applied after the characteristic equation is somewhat simplified by assuming complete symmetry in the system, i.e., $m_1 = m_2$, $c_1 = c_2 = c$, $k_1 = k_2$, $I_{y_1} = I_{y_2} = I$, $\rho_1 = \rho_2 = \rho$, but I_{x_1} not necessarily equal to I_{y_1}, etc. Following that assumption: $a_{11} = a_{17}$, $a_{10} = a_{16}$, $a_7 = a_{13}$, $a_8 = a_{14}$, $a_9 = a_{15}$, and $a_2 = a_3$, and letting $B = a_6 + (a_4^2/a_1) + a_{18}$ and $D = a_{12} + a_7 - 2a_2a_{10}/a_1$, the determinant of the linearized system matrix can be reduced to

$$[(a_{12} - a_{17})\lambda^2 - a_8\lambda - (a_9 - a_6a_{12})] \times \{2[a_{11}\lambda - (a_4a_{10}/a_1)\lambda](a_2a_4/a_1)\lambda - (\lambda^2 + a_5\lambda + B) \times [D\lambda^2 + a_8\lambda + a_6a_{12} + a_9]\} = 0 \quad (35)$$

where the roots of the quadratic factor become after substitution,

$$\lambda_{1,2} = \frac{-\bar{k}_1 \mp [\bar{k}_1^2 - 4\{\bar{\rho}\Gamma^2(1 + 2\bar{\rho}) + \bar{c}\}\bar{I}]^{1/2}}{\bar{I}} \quad (36)$$

where

$$\bar{I} = \bar{I}_{y_1} = \bar{I}_{y_2}$$

A resulting necessary, but not sufficient, condition for stability is

$$\bar{k}_1^2 - 4\{\bar{\rho}(\Gamma^2)(1 + 2\bar{\rho}) + \bar{c}\}\bar{I} < 0 \quad (37)$$

Expanding the second factor in Eq. (35) and collecting terms yields:

$$\lambda^4D + \lambda^3(Da_5 + a_8) + \lambda^2[2(a_2a_4/a_1)(a_4a_{10} - a_{11}) + a_6a_{12} + a_9 + a_5a_8 + BD] + \lambda[a_5(a_6a_{12} + a_9) + Ba_8] + B(a_6a_{12} + a_9) = 0 \quad (38)$$

The conditions for each of the terms in Eq. (38) to be acteristic equation. Granted those conditions, the necessary, but not sufficient, condition for stability is satisfied.

The Routh-Hurwitz criterion can now be applied for Eq. (38). Since it was shown that D, the coefficient of λ^4, is positive, the Routhian determinants must all be positive and nonzero. These determinants will now be developed and examined,

$$\Delta_1 = \text{coefficient of } \lambda^3\text{, shown to be positive}$$

$$\Delta_2 = 2(a_2a_4a_5/a_1)(a_4a_{10} - a_{11})D + a_5{}^2a_8\Delta_1 + a_5BD^2 + 2(a_2a_4a_8/a_1)(a_4a_{10} - a_{11}) + a_8(a_6a_{12} + a_9 + a_5a_8) \quad (39)$$

It is apparent that Eq. (39) is positive for the same conditions that were previously derived, i.e., $a_5 > 0$ and $a_8 > 0$.

It can be shown that Δ_3 (explicitly shown in Ref. 11 but too lengthy to warrant showing here) is also positive for the same conditions.

Finally, Δ_4 = (positive constant term) $\times$ Δ_3 which is obviously positive also.

Thus, the conditions for stability for the special case of complete symmetry are

$$\bar{k} \text{ and } \bar{k}_3 \text{ must be positive} \quad (40)$$

$$\bar{k} > \Gamma^2[a_1 - 4\bar{L}^2]/a_1 \quad (41)$$

$$\bar{k}_1{}^2 - 4\{\bar{\rho}\Gamma^2(1 + 2\bar{\rho}) + \bar{c}\}\bar{I} < 0 \quad (42)$$

$$\bar{k}_1 \geq 0 \quad (43)$$

In addition $c_{1,2} > 0$, when $\omega/s = -1$.

For the special case of complete symmetry and zero attachment lengths, the characteristic equation is

$$(a_2\lambda^2 + a_8\lambda + a_9)[(\lambda^2 + a_5\lambda + B) \times (a_2\lambda^2 - 2(a_2{}^2/a_1)\lambda^2 + a_8\lambda + a_9) + 2(a_2{}^2a_4{}^2/a_1{}^2)\lambda^2] = 0$$

yielding, after substitution, the following separable mode,

$$\lambda_{1,2} = [-\bar{k}_1 \pm (\bar{k}_1{}^2 - 4\bar{c}\bar{I})^{1/2}]/2\bar{I}$$

The necessary conditions for stability are

$$\bar{k} > (a_1 - 4)\Gamma^2/a_1 \quad (44)$$

$$\bar{k}_1{}^2 - 4\bar{c}\bar{I} < 0 \quad (45)$$

and $\bar{k}$, and $\bar{k}_3$ must be positive with the same constraint on $\bar{c}_{1,2}$ as before.

Special case of point mass end masses

As a result of this assumption there is only one variational linearized equation of motion for the system, i.e.

$$\bar{l}'' + \bar{k}_3\bar{l}' + [\bar{k} + 3\Gamma^2]\bar{l} = 0$$

Solving the preceding differential equation directly, yields the following condition for stability:

$$\bar{k}_3 > 2(\bar{k} + 3\Gamma^2)^{1/2} \quad (46)$$

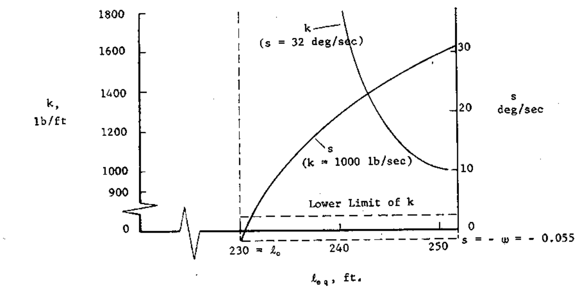

Fig. 2 Variation of cable stiffness k and system spin rate s with length for the equilibrium condition.

Numerical Results

Equilibrium Condition

The equilibrium condition is given by Eq. (27) or, in its dimensional form, by Eq. (21). Solving Eq. (21) for $l_{eq.}$, yields

$$l_{eq.} = [kl_0 + \mu(\rho_1 + \rho_2)(\omega + s)^2]/k - \mu(\omega + s)^2 \quad (47)$$

where l_0 is the unstretched tether system length. In connection with Eq. (47), the following limiting cases have definite physical implications: a) when $k \to \infty$, the cable has infinite stiffness, and with an application of L'Hospital's rule, $l_{eq.} \to l_0$; and, b) a lower limit on k is given by the fact that the cable cannot support the tension generated by the rotational motion unless $k > \mu(\omega + s)^2$.

The following nominal values of the constants are chosen so that an artificial force field of 1 g on the space station is simulated: ω = 0.055°/sec for 2,000-naut mile altitude circular orbit, s = 32°/sec, m_1 = 770 slugs, m_2 = 430 slugs, ρ_1 = 15 ft, ρ_2 = 9 ft, l_0 = 230 ft, k = 1000 lb/ft, and x = 275 ft is the separation distance between the mass centers of the two end bodies. The values selected for the masses correspond to current payload capability of Saturn class launch vehicles for lower Earth orbits. It can be seen that with these parameters the criterion $k > \mu(\omega + s)^2$ is calculated to be $k > 86.94$ lb/ft.

For a steady-state, equilibrium condition, a tensile force of 25,000 lb in the cable is required for the appropriate constants listed previously.

The parameters k and s were then allowed to vary from their nominal values, and the graphs given by Figs. 2 and 3 were obtained. Figure 2 shows that when $k \to \infty$, the value of $l_{eq.}$ approaches l_0 asymptotically. Figure 2 gives a clear indication of the fact that when the system is spinning at a rate equal and opposite to the orbital angular velocity, the value of $l_{eq.}$ is identically equal to l_0 (in the absence of gravitational torque effects).

Finally, Fig. 3 indicates that the equilibrium cable stiffness increases with increasing system rate of spin as indicated in Eq. (21).

Optimization of the Damping System

The characteristic Eq. (31) for the linear system was programed for numerical solution on the IBM 1132 computer. The presence of undamped normal modes of oscillation for the case of complete symmetry between the end bodies, for the case of complete symmetry with zero attachment lengths, and for the case of zero attachment lengths with no rotary and restoring damping was verified.

For the further study of the case of complete symmetry the following parameters were held constant: ω = 0.055°/sec for 2000-naut mile altitude circular orbit, s = 32°/sec, m_1 = m_2 = 600 slugs, $\rho_1 = \rho_2$ = 12 ft, k_3 = 56.70 lb-sec/rad, I_{y_1} = I_{y_2} = 86,400 slug-ft², l_0 = 230 ft.

The damping system was optimized by a numerical search for the rotary spring constants and the rotary damping constants which yielded the minimum time constant, τ (i.e.,

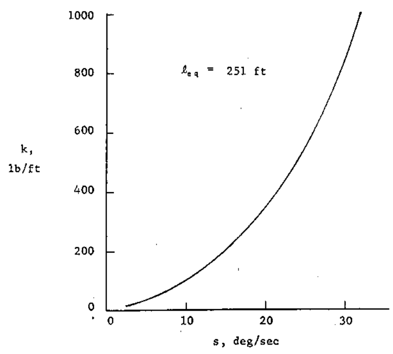

Fig. 3 Variation of cable stiffness k with system spin rate s for the equilibrium condition.

time to reach $1/e$ of the initial amplitude) of the least damped mode of oscillation. This was accomplished by repeated examination of the roots of the system characteristic equation for the case of complete symmetry.

In the preceding section it was indicated that the system is stable for $c > 0$ ($c = c_1 = c_2$). For $c = 0$, the system is unstable if, in addition, $\Gamma = 0$ (or $\omega/s = -1$). The latter statement was verified by the computer results. Figure 4 shows that the optimum value is between 3000 and 5000 ft-lb/rad. The value of 5000 ft-lb/rad was selected for further studies. Stability also requires $k_{1,2} > 0$ ($k_1 = k_2$). Figure 4 shows that the optimum value is near 13,000 ft-lb-sec/rad.

For the study of the general nonsymmetrical case, the following parameters were considered to be nominal: $\omega = 0.055°/\text{sec}$ for 2000 naut miles altitude circular orbit, $s = 32°/\text{sec}$, $m_1 = 770$ slugs, $m_2 = 430$ slugs, $\rho_1 = 15$ ft, $\rho_2 = 9$ ft, $I_{y_1} = 173{,}250$ slug-ft², $I_{y_2} = 34{,}830$ slug-ft², $l_0 = 230$ ft. The damping system design was optimized following the same procedure as in the case of complete symmetry.

Figure 5 shows that the optimum values are $c_{1,2} = 5000$ ft-lb/rad, $k \to k_{\min} = 86.94$ lb/ft, $k_{1,2} = 12{,}000$ ft-lb-sec/rad, and $k_3 = 57$ lb-sec/ft. However, with respect to the cable elastic constant k we note that for $k < 800$ lb/ft, the extension of the cable, from the unstressed value $l_{\text{eq.}} - l_0$, increases significantly (see Fig. 2). Because the probability of cable wraparound increases as k approaches its limiting value, a value of 1000 lb/ft is selected for further studies.

The similarities of the curves $c_{1,2}$ and $k_{1,2}$ to the ones obtained for the case of complete symmetry is obvious. The optimum value of k_3 is close to the value of 56.7 lb-sec/ft initially selected for the damping constant.

Quantitative Effects of Gravity-Gradient Torques

The maximum effect of gravity-gradient torques on the station-cable-counterweight system was evaluated using the equations developed by Thomson.[12] The maximum component of gravitational torques as applied to this particular system is

$$L_{\max} = (3GM_E/2R^3)(I_1 - I_2) \qquad (48)$$

where R is the distance from the center of Earth, I_1 is the moment of inertia about an axis through the system mass center, and normal to the orbital plane, and I_2 is the moment of inertia about the centerline of the system.

Substituting the parameters for the general case into Eq. (48) yields $L_{\max} = 11$ slugs-ft²/sec².

The effect of the spin axis drift attributed to the gravity-gradient torque is given by

$$\alpha_{\max} = L_{\max}/I_1 = 0.0001°/\text{sec}^2$$

This acceleration will give rise to a maximum drift rate of less than 0.0001°/sec, since the gravitational torque is of a predominantly cyclic nature depending on the attitude. This value is considerably smaller than the nominal s of 32°/sec.

The complete stability analysis of this system with the presence of gravity-gradient torques is beyond the scope of the current investigation. Such a stability analysis could be accomplished with the application of Floquet Theory to consider the stability of a linear system with periodic coefficients such as treated by Kane[13] and his students.

Possible Implementation of Cable Requirements

A steel cable with a working stress of 150,000 psi and a 0.5 in.² cross-sectional area could very well support the 25,000-lb equilibrium tension while it satisfies the elasticity requirements (modulus of elasticity, 30×10^6 psi). The weight of a 230-ft length cable is 400 lb, or 1% of the total system weight. It should also be noted that the coefficient of thermal expansion for the steel cable in question is 7.5×10^{-6} in./in./°F.[14]

Conclusions and Recommendations

Based upon the development of the two dimensional equations of motion and various stability criteria, coupled with numerical results, the following conclusions can be made: 1) although the system remains stable in the absence of rotational damping (of the end body motions) such damping is required to obtain reasonable time constants for the nominal parameters considered here; 2) the minimum time constant is not sensitive to changes in the amount of cable damping in the vicinity of the optimum damping constant; 3) the effect of the system spin axis drift attributed to the gravity-gradient torque is small; 4) based on a preliminary investigation of possible cable materials, it appears that the design of a cable which could a) support the equilibrium tension, b) satisfy the elasticity requirements, and c) comprise only a small percentage of the total system weight, appears to be feasible.

In order to guarantee the feasibility of the station-cable-counterweight as an operating satellite system, it is recommended that the present work be extended as follows: a) in the area of stability analysis: extend the problem to a three dimensional model and include the more complete effects of gravity-gradient torques; b) in the area of dynamics: develop the complete three dimensional nonlinear equations of motion including significant external perturbations (e.g. solar pressure) and examine cable mass effects; c) in the area of mechanical implementations: study possible methods of implementing the cable system and rotational damping requirements and investigate the effects of environment (thermal shrinkage, zero "g" etc.) on all components of the system.

Appendix

The constant coefficients (a_1 through a_{18}) used in the linearized equations of motion are

$$a_1 = [\bar{I}_{y_1} + \bar{I}_{y_2} + (1 + \bar{\rho}_1 + \bar{\rho}_2)^2],\ a_2 = \bar{I}_{y_1}$$

$$a_3 = \bar{I}_{y_2},\ a_4 = 2\left(\frac{\omega}{s} + 1\right)(1 + \bar{\rho}_1 + \bar{\rho}_2)$$

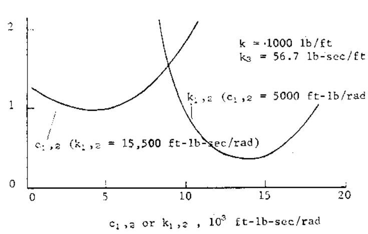

Fig. 4 Optimization of the rotational restoring ($c_{1,2}$) and damping ($k_{1,2}$) constants for the case of complete symmetry.

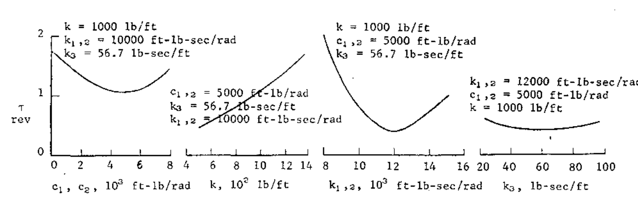

Fig. 5 Optimization of the rotational restoring constant $c_{1,2}$, the cable elastic constant k, the rotational damping constant $k_{1,2}$, and the cable damping constant k_3 for the general case.

$$a_5 = \bar{k}_3,\ a_6 = -\left(\frac{\omega}{s} + 1\right)^2,\ a_7 = \bar{I}_{y_1} + \bar{\rho}_1{}^2$$

$$a_8 = \bar{k}_1,\ a_9 = \left[\bar{\rho}_1\left(\frac{\omega}{s} + 1\right)^2 (1 + \bar{\rho}_2) + \bar{c}_1\right]$$

$$a_{10} = [\bar{I}_{y_1} + \bar{\rho}_1(1 + \bar{\rho}_1 + \bar{\rho}_2)],\ a_{11} = 2\left(\frac{\omega}{s} + 1\right)\bar{\rho}_1$$

$$a_{12} = \bar{\rho}_1\bar{\rho}_2,\ a_{13} = \bar{I}_{y_2} + \bar{\rho}_2{}^2,\ a_{14} = \bar{k}_2$$

$$a_{15} = \left[\bar{\rho}_2\left(\frac{\omega}{s} + 1\right)^2 (1 + \bar{\rho}_1) + \bar{c}_2\right]$$

$$a_{16} = [\bar{I}_{y_2} + \bar{\rho}_2(1 + \bar{\rho}_1 + \bar{\rho}_2)],\ a_{17} = 2\left(\frac{\omega}{s} + 1\right)\bar{\rho}_2$$

$$a_{18} = \bar{k}$$

Nondimensional quantities

$$\bar{I}_{yi} = I_{yi}/\mu l_{\text{eq}}{}^2,\ \bar{\rho}_i = \rho_i/l_{\text{eq.}},\ \bar{k}_i = k_i/\mu l_{\text{eq}}\, s$$

$$\bar{c}_i = c_i/\mu l_{\text{eq.}} s^2,\ i = 1,2$$

$$\bar{k} = k/\mu s^2$$

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